ON CERTAIN INTEGRAL AND HARMONIC FUNCTIONS

A STUDY IN MINIMUM MODULUS

BY

BO KJELLBERG

UPPSALA 1948
APPELBERGS BOKTRYCKERIAKTIEBOLAG

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In publishing this dissertation I express my deep gratitude to my distinguished teacher, Professor ARNE BEURLING, for suggesting my subject and for his unsparingly given interest and valuable advice.

Upsala in May 1948.

Bo Kjellberg

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INTRODUCTION

The discovery of WIMAN that the minimum modulus $m(r)$ of integral functions of order $< \frac{1}{2}$ had the same order of magnitude as the maximum modulus for some r's $\to \infty$, made the former quantity especially interesting. Numerous authors have later studied the set of r-values where $m(r)$ is large. However, due to the considerable difficulties encountered, our knowledge about the minimum modulus is still incomplete; therefore, in the present dissertation, we have found it appropriate to take up the subject anew.

First we state some relevant theorems and then pass on to determine their precision by constructing suitable integral functions. In so doing we particularly find that, unlike what seems to be the general conception, integral functions of order $< \frac{1}{2}$ exist for which $m(r) \to 0$ as $r \to \infty$, save in small exceptional intervals.

It develops that these problems regarding integral functions are closely associated with the question of the growth towards infinity, in the vicinity of a given point on the boundary of an infinitely connected domain, of positive harmonic functions which converge towards zero in the vicinity of all other boundary points. This problem, hitherto apparently untreated, gives rise to a classification of boundary points, a subject touched upon in the final chapter.

CHAPTER I

AN OLD PROBLEM

1. DEFINITIONS. — We denote the minimum modulus of an integral function $f(z)$ by

$$m(r) = \min_{|z|=r} |f(z)|$$

and the maximum modulus by

$$M(r) = \max_{|z|=r} |f(z)|.$$

As usual we characterize the growth of $M(r)$ by saying that $f(z)$ is of finite order ϱ if

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r}$$

is finite and $= \varrho$. Having a function of order ϱ, we obtain a finer distinction by considering "the type"

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\varrho}}$$

and using the terms maximal, mean and minimal type according as this limit is infinite, positive or zero. In this last case, a further classification is made by stating that a function of minimal type is of convergence or divergence type as

$$\int_1^{\infty} \log M(r) \frac{dr}{r^{1+\varrho}}$$

is convergent or divergent.

In accordance with Borel and Valiron, we say that a function of order ϱ is of regular growth if

$$\lim_{r\to\infty} \frac{\log \log M(r)}{\log r}$$

does exist and, consequently, is $= \varrho$. A function of regular growth and mean type of order ϱ is said to be of very regular growth if both

$$\overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\varrho} = A \text{ and } \underline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\varrho} = B$$

are positive. If here $A = B$, we call $f(z)$ a function of perfectly regular growth.

For functions of irregular growth one may conveniently define "the lower order" λ as

$$\lambda = \underline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r}.$$

2. THE CLASS OF INTEGRAL FUNCTIONS CONSIDERED. — Here we are mainly concerned with integral functions, the order of which lies in the interval $0 < \varrho < \frac{1}{2}$. According to classical theorems, such a function may be represented as a WEIERSTRASSIAN canonical product of the simplest kind

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right), \tag{1}$$

taking $f(0) = 1$.

We shall pay attention to the set of values of r, where $m(r)$ takes on comparatively large values, e. g. $m(r) >$ constant or $>$ an increasing function of r. Fixing the moduli of the zeros, we observe by a glance at (1) that the most unfavourable case will occur when the zeros are projected on a radius from the origin, the negative real axis, for instance. Then, $m(r)$ will evidently be as small as possible and $M(r)$ as large as possible. It is, therefore, sufficient to study "oriented" integral functions:

$$f(z) = \prod_1^\infty \left(1 + \frac{z}{r_n}\right), \quad r_n = |a_n|. \tag{2}$$

Order and type-class are unchanged.

3. LINEAR AND LOGARITHMIC DENSITIES. — We shall use such measures for the purpose of describing, somewhat precisely, certain sets of r-values. Given such a set E, then

$E(1, R)$ may be the part of E situated in $(1, R)$. Dividing the linear and logarithmic measures

$$m\,E(1, R) = \int_{E(1, R)} dr$$

and

$$m_l\,E(1, R) = \int_{E(1, R)} d\,\log r$$

by $R-1$ and $\log R$ respectively, we will obtain the respective mean densities in $(1, R)$. The upper and lower linear and logarithmic densities of E are then defined as upper and lower limits of these mean densities when R tends to infinity. We will examine the connection between these densities $\overline{\text{dens.}}\ E$, $\underline{\text{dens.}}\ E$, $\overline{\text{log. dens.}}\ E$ and $\underline{\text{log. dens.}}\ E$. By definition:

$$m_l\,E(1, R) = \int_{E(1, R)} \frac{dr}{r} = \int_1^R \frac{d\,m\,E(1, r)}{r}$$

and after integration by parts

$$m_l\,E(1, R) = \frac{m\,E(1, R)}{R} + \int_1^R \frac{m\,E(1, r)}{r^2}\,dr. \tag{3}$$

On some reflection we may summarize:

$$1 \geq \overline{\text{dens.}}\ E \geq \overline{\text{log. dens.}}\ E \geq \underline{\text{log. dens.}}\ E \geq \underline{\text{dens.}}\ E \geq 0. \tag{4}$$

If, therefore, mean densities of finite intervals are considered, then the logarithmic density may generally be expected to oscillate less than the linear one as $R \to \infty$.

4. REVIEW OF PREVIOUS INVESTIGATIONS. — In a paper, published in 1903, WIMAN studied some special integral functions which, according to VALIRON's terminology, mentioned above, are of perfectly regular growth, e. g.

$$f(z) = \prod_1^\infty \left(1 + \frac{z}{n^\alpha}\right), \quad \alpha = \frac{1}{\varrho} > 1\,. \tag{5}$$

He found

$$\log|f(re^{i\varphi})| = \frac{\pi r^{\varrho}}{\sin \pi\varrho}\cos\varrho\varphi - \frac{1}{2\varrho}\log 2\pi r^{\varrho} + O\left(\log\frac{\delta}{r}\right) \tag{6}$$

where δ signifies the distance from $z = re^{i\varphi}$ to the nearest zero $-n^{\alpha}$. WIMAN here observed that throughout the whole plane, apart from small circles around the zeros, the following is true as soon as $\varrho < \frac{1}{2}$.

$$|f(z)| > e^{|z|^{\varrho-\varepsilon}}, \quad \varepsilon > 0. \tag{7}$$

He now set up as a hypothesis that for every integral function of order $\varrho < \frac{1}{2}$

$$m(r) > e^{r^{\varrho-\varepsilon}}, \tag{8}$$

holds for a sequence of arbitrarily large values of r.

In the next year a PHRAGMÉN work appeared, in which the PHRAGMÉN-LINDELÖF principle, later on so famous, made its entrance into mathematics, though in somewhat incomplete form.

By aid of the PHRAGMÉN result WIMAN could prove his conjecture in 1905. Evidently his theorem created great interest and stimulated also other function theorists to further investigations, among others LINDELÖF and LITTLEWOOD. Within a decade VALIRON and WIMAN had independently found proofs of the following LITTLEWOOD hypothesis:

$$\log m(r) > (\cos \pi\varrho - \varepsilon)\log M(r), \tag{9}$$

for a sequence of indefinitely increasing values of r. In the special case (6) we may observe the presence of just such a factor $\cos \pi\varrho - \varepsilon$.

By means of elementary estimations AMIRÀ (1925) secured the validity of (9) in intervals $R_n \leq r \leq R_n + R_n^{1-\varrho-\varepsilon}$, $R_n \to \infty$, i. e. $|f(z)|$ is certainly large in a sequence of annular regions.

Some formulae in a work of PÓLYA (1926) will be given later.

In 1930, VALIRON could by a special method with complex integration prove the above results and also

$$\log m(r) > (\cos \pi\varrho - \varepsilon)\log M(r) > r^{\varrho-\varepsilon}, \quad \varrho < \tfrac{1}{2}, \tag{10}$$

for a sequence of values of $r \to \infty$.

5. LATER RESULTS. — A decisive step forward was to be noted in 1927 when Besicovitch published sharp results of great interest. He succeeded in estimating linear densities of some sets E of r-values, corresponding to large $m(r)$:

(11) $\overline{\text{dens.}}\ E\{\log m(r) > \cos \pi\varrho' \log M(r)\} \geq 1 - \frac{\varrho}{\varrho'},\ 0 \leq \varrho < \varrho' < 1;$

(12) $\overline{\text{dens.}}\ E\{\log m(r) > r^{\varrho-\varepsilon}\} \geq 1 - 2\varrho,\ 0 < \varrho < \tfrac{1}{2},\ \varepsilon > 0.$

His method of proof was quite elementary: by suitable and allowable displacements of zeros, it was in certain intervals possible to compare the given $f(z)$ with functions of perfectly regular growth, such as (5). By an oversight, the proof of the relation (12) — being here most interesting — was valid only for functions of regular growth.

This restriction was removed by Pennycuick (1935), the set of r:s satisfying (10) has an upper linear density $\geq 1 - 2\varrho$.

A formula of Denjoy (1931) will be treated in next chapter.

Whittaker (1933) observed that a Wiman theorem remains valid even for those integral functions of order $\geq \frac{1}{2}$, the lower order of which $\lambda < \frac{1}{2}$. We shall return to this question.

The dissertation of Beurling (1933) gives, in its richness of results, also new possibilities for investigation of the minimum modulus, which we shall make use of further on.

Here we have mentioned only important general advances. The bibliography will give a more complete picture. However, no particular attention has been paid to numerous works, where the problem is simplified by introduction of special assumptions with respect to the regularity of the growth of functions or, for instance, by its restriction to the case $\varrho = 0$.

A type of question closely connected with the problem of the minimum modulus is the following:

An integral function $f(z)$ of at most order ϱ is bounded on a certain sequence of points $a_1, a_2, a_3, \ldots, a_n, \ldots \to \infty$. When does this imply $f(z) \equiv$ constant?

As such questions seem to have been treated only under very special assumptions with regard to the sequence $\{a_n\}$, we will not review the subject here, but return to it in Chap. IV.

CHAPTER II

SOME SIMPLE METHODS FOR THE STUDY OF MINIMUM MODULUS

6. AN IMPROVEMENT OF A CLASSICAL THEOREM. — One of the most important results of the PHRAGMÉN-LINDELÖF standard work (Acta Math. 1908) may, if somewhat simplified, be expressed as follows:

"*If an analytic function* $f(z)$ *is regular in an angular region* D *with angle* $\frac{\pi}{\varrho}$, *and of at most order* ϱ, *minimal type, as* $z \to \infty$ *in* D, *then* $|f(z)| \leq C$ *on the two boundary radii implies* $|f(z)| \leq C$ *throughout* D."

As is easily seen this at once gives the WIMAN theorem (8) for integral functions up to order $\frac{1}{2}$, minimal type.

The PHRAGMÉN-LINDELÖF theorem has later been variously generalized. One fact that immediately appears upon inspection of the proof of the theorem is that the condition "minimal type of order ϱ" may be replaced by

$$\lim_{r \to \infty} \frac{\log M(r)}{r^{\varrho}} = 0,$$

of which fact we may avail ourselves later.

In a lecture, Prof. BEURLING has given an improvement especially suitable for treatment of our minimum modulus problem. We here take the liberty of rendering an account of his theorem.

Consider a holomorphic function $f(z)$ in a region D between two straight lines making an angle $\frac{\pi}{\sigma}$ at the origin and having its bisector along the positive real axis.

We suppose

1° $|f(r e^{\pm i \frac{\pi}{2\sigma}})| \leq \varphi(r)$, where $\varphi(r)$ is a continuous function such that

$$\overline{\lim_{r\to\infty}} \frac{\log \varphi(r)}{r^{\varrho}} = 0,$$

where ϱ is a number in the interval $0 < \varrho < \sigma$,

$$2^{\circ} \qquad \lim_{r\to\infty} \frac{\log M(r)}{r^{\varrho}} = 0,$$

but, on the other hand, $f(z)$ is supposed not to be bounded in D.

Then the inequality

$$\log M(r) < \frac{\log \varphi(r)}{\cos \frac{\pi\varrho}{2\sigma}} \tag{13}$$

holds true for a sequence $r = r_n \to \infty$.

To prove this, consider the function $f(z)\, e^{-\varepsilon z^{\varrho} - a}$, which is regular in D (we choose a branch of z^{ϱ}, which is >0 when $z>0$). On the boundary

$$\log |f(z)\, e^{-\varepsilon z^{\varrho} - a}| \leq \log \varphi(r) - \varepsilon \cos \frac{\pi\varrho}{2\sigma} r^{\varrho} - a.$$

According to the supposition, we certainly have, for every $\varepsilon > 0$ and for every real constant a,

$$\log \varphi(r) < \varepsilon \cos \frac{\pi\varrho}{2\sigma} r^{\varrho} + a$$

as soon as r is sufficiently large. If a is large enough, the inequality is valid for every r. We then choose $a(\varepsilon)$ such that

$$\log \varphi(r) \leq \varepsilon \cos \frac{\pi\varrho}{2\sigma} r^{\varrho} + a(\varepsilon) \tag{14}$$

holds with equality for at least one $r = r(\varepsilon)$. Of course, $a(\varepsilon)$ and $r(\varepsilon) \to \infty$ as $\varepsilon \to +0$ (φ cannot be bounded!). Now the modulus $|f(z)\, e^{-\varepsilon z^{\varrho} - a(\varepsilon)}| \leq 1$ on the radii $\arg z = \pm \frac{\pi}{2\sigma}$ and thus, by the PHRAGMÉN-LINDELÖF theorem, it is ≤ 1 at any point of D. Hence we have for $r = r(\varepsilon)$, simultaneously,

$$\left.\begin{aligned} \log \varphi(r) &= \varepsilon \cos \frac{\pi\varrho}{2\sigma} r^{\varrho} + a(\varepsilon) \\ \log M(r) &\leq \varepsilon r^{\varrho} + a(\varepsilon) \end{aligned}\right\},$$

which gives

$$\log \varphi(r) \geq \cos \frac{\pi\varrho}{2\sigma} \log M(r) + a(\varepsilon)\left(1 - \cos\frac{\pi\varrho}{2\sigma}\right).$$

This is a little more than (13), as $a(\varepsilon) \to \infty$ when $\varepsilon \to +0$.

We can now improve the LITTLEWOOD inequality (9).

Let the region D be the whole plane, cut along the negative real axis, for instance. It follows from Chap. I that it is sufficient to consider integral functions with zeros on this line in order to obtain this conclusion from (13), taking $\varphi(r) = m(r)$:

For every integral function of at most order ϱ, minimal type, $0 < \varrho < \frac{1}{2}$,

$$\log m(r) > \cos \pi\varrho \log M(r) \tag{15}$$

holds true for a sequence $r = r_n \to \infty$.

BEURLING's theorem (13) may be expressed otherwise. Here we may introduce the least majorant φ^* of φ, having the property that $\log \varphi^*(r)$ is a concave function of r^{ϱ}. From the above proof it then follows that

$$\log M(r) < \frac{\log \varphi^*(r)}{\cos \frac{\pi\varrho}{2\sigma}} \tag{16}$$

is valid for *all* $r >$ an $r_0 \geq 0$.

Now it is not difficult to see that by this method we also obtain a theorem corresponding to VALIRON's double inequality (10).

7. SOME COMMENTS ON DENJOY'S FORMULA. — PÓLYA started from the integrals ($r_n > 0$, $0 < \sigma < 1$):

$$\int_0^{\infty} \frac{\log\left(1 + \frac{r}{r_n}\right)}{r^{1+\sigma}}\, dr = \frac{\pi}{\sigma \sin \pi\sigma} \cdot \frac{1}{r_n^{\sigma}} \tag{17}$$

$$\int_0^{\infty} \frac{\log\left|1 - \frac{r}{r_n}\right|}{r^{1+\sigma}}\, dr = \frac{\pi}{\sigma} \cot \pi\sigma \cdot \frac{1}{r_n^{\sigma}}$$

If $-r_1, -r_2, \ldots, -r_n, \ldots$ are regarded as zeros of an integral function $f(z)$ and if $\sum_1^\infty \frac{1}{r_n^\sigma}$ is convergent, one obtains:

$$\int_0^\infty \frac{\log m(r) - \cos \pi\sigma \log M(r)}{r^{1+\sigma}}\, dr = 0 \tag{18}$$

valid for functions of order $<\sigma$ or $=\sigma$, convergence type.

PÓLYA could prove, although not quite elementarily, that the integrated function certainly must be positive for a sequence of arbitrarily large r-values, which constituted a new proof of WIMAN's theorem in version (9).

The property mentioned of the numerator of (18) turned quite obvious through an ingenious device of DENJOY. In a short paper — which has apparently received very little attention — he gave an entirely elementary proof of this striking relation:

$$\int_R^\infty \frac{\log m(r) - \cos \pi\sigma \log M(r)}{r^{1+\sigma}}\, dr > k(\sigma)\, \frac{\log M(R)}{R^\sigma}, \tag{19}$$

valid for every $R>0$; the constant $k(\sigma)$ is >0 and independent of R.

We will sketch the simple proof. Putting

$$\psi(x) \equiv \int_x^\infty \frac{\log |1-x| - \cos \pi\sigma \log (1+x)}{x^{1+\sigma}}\, dx,$$

it is plain by (17) that $\psi(0) = \psi(\infty) = 0$. Then a constant $k(\sigma)$, as large as possible, is to be determined such that

$$d(x) \equiv \psi(x) - k(\sigma)\, \frac{\log (1+x)}{x^\sigma} > 0 \tag{20}$$

for $0<x<\infty$; at the end-points $d(0) = d(\infty) = 0$. If we require $d(x)$ first to be steadily increasing and then steadily decreasing, (20) certainly holds good. In this manner we obtain

$$k(\sigma) = \frac{1 - \sin \pi \left| \frac{1}{2} - \sigma \right|}{\frac{1}{2} - \left| \frac{1}{2} - \sigma \right|}$$

which really is positive. From (20) we then get (19) by means of obvious transformations and summation.

One may especially observe the neat formula for $\sigma = \frac{1}{2}$:

$$\int_R^\infty \frac{\log m(r)}{r^{3/2}} \, dr > 2 \frac{\log M(R)}{\sqrt{R}}, \tag{21}$$

which holds true for every integral function of at most order $\frac{1}{2}$, convergence type.

By simply stating that the right-hand member of (19) is positive we get PÓLYA's result:

For every integral function of at most convergence type of order ϱ, $0 < \varrho < 1$, *it is true that*

$$\log m(r) > \cos \pi\varrho \log M(r) \tag{22}$$

for a sequence of arbitrarily large values of r.

If the order of magnitude of $\frac{\log M(R)}{R^\varrho}$ also is taken into consideration, a single glance at (19) is sufficient to realize that

$$\log m(r) > \cos \pi\varrho \log M(r) > r^{\varrho - \varepsilon}, \quad 0 < \varrho < \frac{1}{2}, \tag{23}$$

applies to functions of order $> \varrho - \varepsilon$, but at most $= \varrho$, convergence type.

However, (16) gives a sharper statement with "minimal type" in place of "convergence type".

By means of a quantitative estimation of the integral in (19) the validity of the above-mentioned inequalities in annulae with breadth $R^{1-\varrho-\varepsilon}$ may be shown — in other words, a generalization of AMIRÀ's result. Finally, theorems with linear densities may be obtained — but not as precise as those of BESICOVITCH.

Thus DENJOY's simple formula renders most of the results arrived at so far in this field.

8. LOGARITHMIC DENSITIES, INTRODUCED BY MEANS OF BEURLING THEOREMS. — Using somewhat simplified assumptions we first give a lemma from BEURLING's dissertation (see his formulae nos. (41) and (75)!).

Let D denote a multiply connected domain within the circle C_R (i. e. $|z|=R$) which is bounded by the circle and a finite number of continua, collectively termed Γ. The circular projection of Γ on the positive real axis, E, signifies the set of points passed by $|z|$, as z passes through Γ. Let Δ denote the domain obtained when the circular disk $|z|<R$ is slit along E.

The harmonic measures $\omega(z, \Gamma, D)$ and $\omega(z, E, \Delta)$ are defined as the functions, harmonic in D and Δ respectively, which take the boundary value 0 on Γ and E respectively, but $\to 1$ as $|z|\to R$. As before $m_l E(r, R)$ denotes the logarithmic length, $\int d \log r$ over E in (r, R).

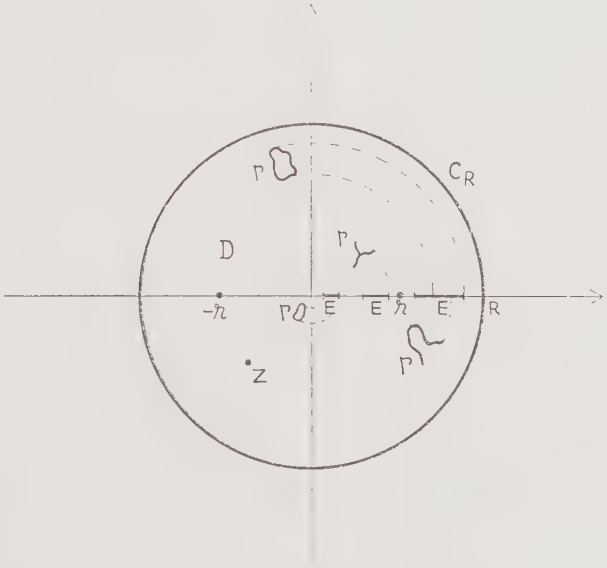

Fig. 1.

Then

$$\omega(z, \Gamma, D) \leq \omega(-r, E, \Delta) < 2\, e^{-\frac{1}{2} m_l E(r, R)} \tag{24}$$

holds true for every z in D, $r=|z|$.

Making use of the principle of the harmonic majorant this immediately leads to (BEURLING p. 96):

Let $f(z)$ be holomorphic for $|z|<R$; let $0\leq r_1<r_2\leq R$ and $\mu>0$ a number less than $M(r_1)$. The set in (r_1, r_2) where $m(r)\leq\mu$ may be denoted by $E(r_1, r_2)$. Then

$$(25)\quad m_l E(r_1, r_2)<\log 4+2\left\{\log\log\frac{M(r_2)}{\mu}-\log\log\frac{M(r_1)}{\mu}\right\}$$

is true.

We are now in a position to pass on to applications.

Let $f(z)$ be an integral function of order ϱ and $C>0$ a constant, however large. We choose an r_1 so that $M(r_1)>C$ and apply (25). After division by $\log r_2$ we then in the limit find that the upper logarithmic density of the set of r-values in $(1, \infty)$ for which $m(r)\leq C$ definitely is $\leq 2\varrho$.

We prefer to express the result in the following theorem:

For a non-constant integral function $f(z)$ of order ϱ

$$(26)\qquad \underline{\log.\ \text{dens.}}\ E\{m(r)>C\}\geq 1-2\varrho$$

holds true, where $C>0$ is an arbitrary constant.

Thus we have, for functions of order $0\leq\varrho<\frac{1}{2}$, obtained a remarkably precise statement. Linear densities can, on the other hand, not be advantageously used in this connection. For every such ϱ, integral functions exist of which the $\underline{\text{dens.}}\ E\{m(r)>C\}=0$, as is easily found by constructing illustrative examples (cf. e. g. BESICOVITCH); we would also bring to mind WIMAN's example (5) where $\underline{\text{dens.}}\ E\{m(r)>C\}=1$.

Formula (25) may also be used with advantage in the investigation of integral functions about which is known only that the lower order λ lies in the interval $0\leq\lambda<\frac{1}{2}$. All that is necessary is to choose a sequence of intervals (r_1, r_2), (r_2, r_3), $\ldots$, (r_n, r_{n+1}), $\ldots$, where $r_{n+1}=r_n^{1+\varepsilon}$, $\varepsilon>0$, and to apply (25) in every interval, putting $\log\mu=r_n^{\lambda-\varepsilon}$. For certain indices n_ν, according to the definition of λ, $\log M(r_{n_\nu})<r_{n_\nu}^{\lambda+\varepsilon}$. After adding these inequalities up to such indices n_ν, dividing by $\log r_{n_\nu}$ and letting $n_\nu\to\infty$ the following theorem is obtained:

For a non-constant integral function with the lower order λ *in the interval* $0 \leq \lambda < \frac{1}{2}$

$$\overline{\text{log. dens.}}\ E\{\log m(r) > r^{\lambda-\varepsilon}\} \geq 1 - 2\lambda \tag{27}$$

for every $\varepsilon > 0$.

We would finally point out that both Besicovitch's theorem (12) for $0 \leq \varrho < \frac{1}{2}$ and (8) for $\varrho = \frac{1}{2}$, minimal type, are obtained by applying (25) to intervals $(r, r^{1+\varepsilon})$ where $M(r)$ takes values touching the largest possible for the order in question (the connection between logarithmic and linear measures having been indicated in Chap. I).

Here — as hitherto in this chapter — we do not carry the method as far as possible but stop at the rapidly derived formulae (26) and (27). It is our main intention to focus interest on how the study of apparently quite different problems pertaining to harmonic functions can well make available new information about the questions applying to integral functions: e. g. as regards the precision of the theorems just proved.

CHAPTER III

ON POSITIVE HARMONIC FUNCTIONS IN SOME DOMAINS OF INFINITE CONNECTIVITY

9. QUESTIONS OF EXISTENCE. — In the whole of this chapter we deal with domains determined by pairs of numbers a and b which satisfy the condition $1<a<b$. We slit the complex plane along the intervals (b^n, ab^n), $n=0, \pm 1, \pm 2, \ldots$, of the positive real axis. The domain in question $D = D(a, b)$ will then be the entire plane outside the slits $E = E(a, b)$ just defined.

We will then investigate functions $u(z)$ satisfying the following conditions:

1° *$u(z)$ is harmonic and positive in D,*

2° *$u(z)$ converges to zero in the vicinity of every finite boundary point.*

The class comprising these functions we will call U.

Obviously such functions cannot be bounded in the vicinity of the boundary point $z=\infty$.

That this class U of harmonic functions is not void may be shown in, for instance, the following manner:

Choose a sequence of points $-2 \geq x_1 > x_2 > \ldots > x_n > \ldots \to -\infty$.

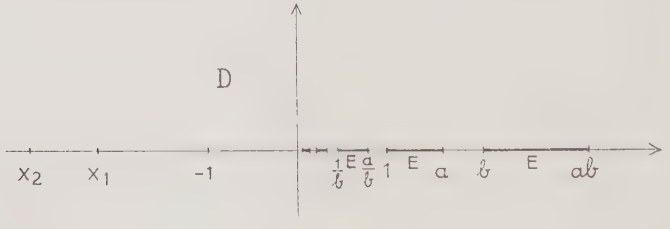

Fig. 2.

We now intend to form the GREEN's functions for the domains: $G(z, x_n)$, singular at x_n. First, however, we should make sure that the two critical boundary points 0 and ∞ in accor-

dance with LEBESGUE's terminology also are "regular for DIRICHLET's problem". Here we might advantageously apply a criterium given in BEURLING's dissertation (p. 66) according to which a boundary point $z' = z_0$ always is regular if the variation of $\log|z' - z_0|$ is unbounded as z' passes the part of the boundary located in a circle $|z - z_0| < \varepsilon$, ε however small. In the present case the origin is a limiting point of E-intervals, each being of the logarithmic length $\log a$. Thus the regularity is assured. The same holds for $z = \infty$ as is clearly seen e. g. by inversion in a circle.

Hence we may start with a sequence $G(z, x_n)$ satisfying the usual conditions, i. e. $G(z, x_n) + \log|z - x_n|$ is harmonic in D and $G(z, x_n)$ tends to zero in the vicinity of *every* boundary point.

Now putting

$$h_n(z) \equiv \frac{G(z, x_n)}{G(-1, x_n)},$$

then for $n = 1, 2, 3, \ldots$:

1° $h_n(z)$ is harmonic in D except at $z = x_n$,
2° $h_n(z) > 0$ in D, $= 0$ on the boundary,
3° $h_n(-1) = +1$.

We would here bring to mind a well-known formula. If $u(z)$ is a function which is harmonic and positive in a circle of radius R, there is the following relationship between the values of the function u_0 at the centre of the circle and u_1 at a point at distance $r < R$ from the centre:

$$u_0 \frac{R - r}{R + r} \leq u_1 < u_0 \frac{R + r}{R - r}. \tag{28}$$

By repeated use of (28), due regard being paid to 3°, it is clear that there is a constant M_1 which is independent of n, such that:

$$h_n(z) \leq M_1 \text{ for } |z| = \frac{1}{2}\left(\frac{a}{b} + 1\right).$$

Within this circle C_1 an absolute majorant of all $h_n(z)$ is, therefore, obtained by taking the harmonic function $H_1(z)$ which is equal to M_1 on C_1 and to 0 on the part of E within C_1. From

the foregoing discussion of the regularity of the boundary points it is evident that this absolute majorant $H_1(z)$ tends to zero both as $z \to$ any point on the slits E within C_1 and as $z \to 0$.

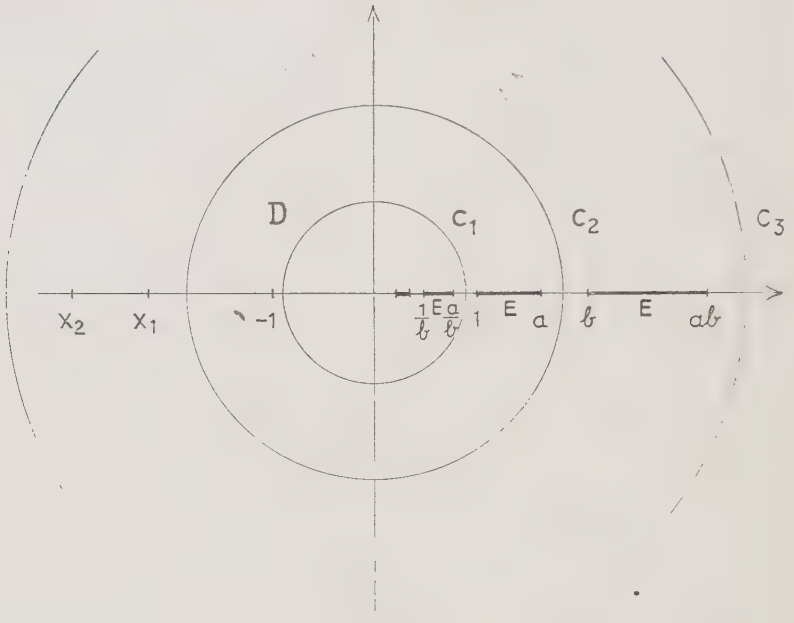

Fig. 3.

There is, however, a continous majorant $H(z)$, vanishing on the boundary, which for every fixed z is majorant of $\{h_n(z)\}$ as soon as n is so large that $|x_n| \geq |z| + 1$. This will be manifest upon applying the above reasoning to large circles C_m with, e. g., radii $\frac{1}{2}(a b^{m-2} + b^{m-1})$. See Fig. 3!

After these preparative steps the proof may be completed. Well-known theorems make it evident that the sequence $h_n(z)$ with the above-mentioned properties is, according to MONTEL, *a normal family*, viz. it is possible in this case to choose a partial sequence which in every closed subdomain Δ of D converges uniformly to a positive harmonic $u(z)$. The above discussion of majorants makes it plain that $u(z)$ tends to zero in the vicinity of *every* finite boundary point. Thus $u(z) \subset U$.

10. A UNIQUENESS THEOREM. — We would now show that $u(z)$ after normalizing $u(-1) = +1$ is unique in the class U.

This is by no means a matter of course; in a subsequent chapter we will discusse the question of uniqueness in general cases.

Suppose that in a domain D of the type in question there were two functions $u(z)$ and $v(z)$, both $\subset U$, and that $u(z) \not\equiv c v(z)$, c being constant. We will show that this assumption leads to an absurdity.

We use the designations in Fig. 3. From (28) follows that there is a constant $k > 0$, such that for z on the circle C_1 it is true that

$$\frac{u_{max}}{u_{min}} < k, \tag{29}$$

and the same applies to $v(z)$.

The sequence of intervals $E = E(a, b)$ is so constructed that it is invariant by the group of transformations of the plane: $T(z) = bz$. Hence the relation (29) is true with the same k on $C_2, C_3, \ldots, C_n, \ldots$ also. If we put

$$\min_{C_n} \frac{u(z)}{v(z)} = k_n, \quad n = 1, 2, 3, \ldots, \tag{30}$$

then there is at least one point z_n on C_n at which

$$u(z_n) - k_n v(z_n) = 0. \tag{31}$$

Since $u(z) - k_{n+1} v(z) \geq 0$ on C_{n+1}, then according to the maximum principle $u(z_n) - k_{n+1} v(z_n) > 0$. This gives

$$k_1 > k_2 > \ldots > k_n > \ldots \to \varkappa > 0. \tag{32}$$

The conclusion $\varkappa > 0$ follows from (29) if due regard be taken to the fact that, for instance, $u(z) - k_1 v(z)$ — which vanishes at some point on C_1 — according to the maximum principle must change signs and therefore also take the value 0 at points on every C_n, $n \geq 2$. Hence

$$\varkappa \geq \frac{k_1}{k^2} > 0.$$

Let us now inspect the function $u(z) - \varkappa v(z)$ also belonging to U!

On C_n it is according to (31) true that

$$u(z_n) - \varkappa v(z_n) = \varepsilon_n v(z_n),$$

where $\varepsilon_n = k_n - \varkappa \to +0$ as $n \to \infty$.

Then by (29) it is true for every z on C_n that

$$0 < u(z) - \varkappa v(z) < k^2 \varepsilon_n v(z),$$

i. e.

$$u(z) - \varkappa v(z) = o\left(v(z)\right) \text{ as } n \to \infty. \tag{33}$$

As has just been observed there are, however, points on every C_n where $u(z) - k_1 v(z) = 0$, or $u(z) - \varkappa v(z) = \varepsilon_1 v(z)$, which is not in agreement with (33).

It has thus been proved that if $u(z)$ and $v(z)$ both belong to U, then

$$\frac{u(z)}{v(z)} \equiv \text{constant}. \tag{34}$$

11. A FUNCTIONAL EQUATION. — From the uniqueness theorem we can draw an important conclusion as to the structure of $u(z)$. As the E-intervals are so constructed that $u(z) \subset U$ leads to $u(bz) \subset U$, then (34) gives the following functional equation:

$$u(bz) \equiv c u(z). \tag{35}$$

By the maximum principle the constant $c > 1$; putting $c = b^\varrho$ we therefore get $\varrho > 0$.

One consequence of (35) is that the values of $u(z)$ in the entire plane are very simply determined by its values in, e. g. the annula $\frac{1}{b} < |z| \leq 1$. We prefer to rewrite the formula

$$u(bz) \equiv b^\varrho u(z) \tag{36}$$

by putting $bz = r e^{i\Theta}$, obtaining

$$u(r, \Theta) = r^\varrho \psi(r, \Theta), \tag{37}$$

where $\psi(r, \Theta) \equiv \psi(br, \Theta)$.

As to the growth of $u(z)$ (37) enables us to infer that $u(z)$ is of mean type of the, as yet, undetermined order $\varrho = \varrho\ (a, b)$; and, in addition, that $u(z)$ is of *very regular growth*, as defined by VALIRON. (Of course, here $u(z)$ corresponds to $\log M(r)$ in the definitions in Chap. I.)

12. SOME SIMPLE FACTS CONCERNING THE GROWTH OF $u(z)$. — From (37) it is easy to obtain an upper limitation of the order ϱ. We cut the circle $|z| \leq R$ along the positive real axis and imagine $u(z)$ represented as a function of its boundary values in this sector. Letting $R \to \infty$ and rejecting the positive contribution of the arc, we get by GREEN's formula for a point $z = -a$ on the negative real axis:

$$u(a, \pi) \geq \frac{\sqrt{a}}{\pi} \int_0^\infty \frac{u(r, 0)\, dr}{\sqrt{r}\,(r+a)}. \tag{38}$$

Thus the integral of the right side must be convergent; which, therefore, also must be true for

$$\sum_0^\infty \int_{ab^n}^{b^{n+1}} \frac{r^\varrho\, \psi(r, 0)}{\sqrt{r}\,(r+1)}\, dr.$$

In the first interval (a, b), $\psi(r, 0)$ is a continuous, positive function; as $\psi(br, 0) \equiv \psi(r, 0)$, the same values recur in the remaining intervals. Hence the series diverges for $\varrho = \frac{1}{2}$, i. e. we must have, for all a, b in question,

$$\varrho(a, b) < \frac{1}{2}. \tag{39}$$

A lower limitation of ϱ is rapidly obtained by means of the BEURLING inequality (24) which also gives (25) with $u(z)$ in place of $\log \frac{|f(z)|}{\mu}$:

$$m_l E(1, R) < \log 4 + 2\varrho \log R.$$

As E here has a logarithmic density $= \frac{\log a}{\log b}$, we can summarize the knowledge about ϱ obtained so far in

$$\frac{\log a}{2 \log b} \leq \varrho(a, b) < \frac{1}{2}. \tag{40}$$

While on the subject we would call attention to some simple conclusions to be drawn from the PHRAGMÉN-LINDELÖF theorem. We have up to the present stated this theorem with special reference to analytic functions $f(z)$ in angular regions. Evidently it is quite permissible to replace $\log |f(z)|$ by a subharmonic function $u(z)$; we also might just as well consider a region which is *part* of the angular region in question.

From the uniqueness theorem we already have that our function $u(z)$ is symmetric with respect to the real axis: $u(z) \equiv u(\bar{z})$.

Take a circle $|z| = R$ and on the arc in the upper half plane take two arbitrary points P and Q; let P be the right-hand point. Denote with L the bisector of the arc PQ dividing the plane into the halves H and H'; H may contain the point P and the positive real axis.

For $z \subset H$ we define $z' \subset H'$ as the symmetric point with respect to L. We will then study the function

$$v(z) \equiv u(z) - u(z'),$$

harmonic in $H - E$, and note that $v(z) \leq 0$ on the boundary $L + E$.

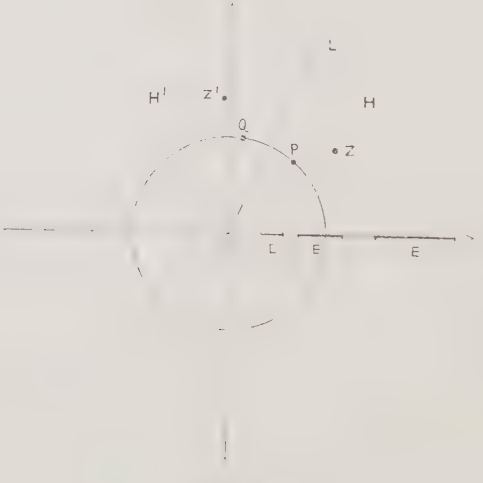

Fig. 4.

As the order is < 1, even $< \frac{1}{2}$, it must be true that $v(z) \leq 0$ in all H; the sign of equality only applies on L. Therefore, in particular

$$u(P) < u(Q);$$

P and Q being arbitrary points on $|z| = R$, we therefore conclude that $u(Re^{i\Theta})$ is steadily increasing as Θ passes from 0 to π; similarly in the lower half plane. Thus we obtain:

On every circle $|z| = R$ the function $u(z)$ attains its minimum for $z = +R$ and its maximum for $z = -R$. Hence

$$\max_{|z| \leq R} u(z) = u(-R). \tag{41}$$

The same simple method is of course useful in drawing many other conclusions as to the growth of $u(z)$; e. g. that $u(z)$ is steadily increasing along every radius in the left half plane.

13. AN AUXILIARY TRANSFORMATION. — For the purpose of later applications we need to know slightly more about ϱ than (40). An exact determination seems to cause difficulties. Therefore we will restrict ourselves to investigating cases when ϱ approaches its upper and lower bounds, which, by the way, are the very cases of interest for our problem.

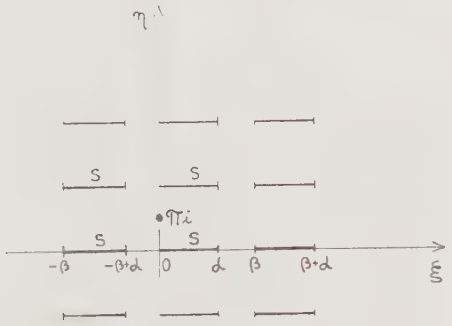

Fig. 5.

For simplicity's sake we will map the domain D in the z-plane on a ζ-plane, slit along segments S, by means of the transformation $\zeta = \log z$. As our E-intervals are of the same logarithmic length, $\log a$, this conformal representation has certain advantages.

Thus $u(z)$ is transformed into a positive, harmonic function $w(\zeta)$ in the described domain Δ; $w(\zeta)$ vanishes on the segments S.

As to $w(\zeta)$ the relation (36) corresponds to

$$w(\zeta + \beta) \equiv e^{\beta\varrho} w(\zeta), \tag{42}$$

where $\beta = \log b$; similarly we put $\alpha = \log a$.

Of course, we also have

$$w(\zeta + 2\pi i) \equiv w(\zeta). \tag{43}$$

The method we will use to estimate the order $\varrho(a, b)$ simply is a detailed study of formula (38) wherein we now — as $\varrho < \frac{1}{2}$ — can put the sign of equality. After the transformation mentioned the cut plane corresponds to the strip $0 \leq \eta \leq 2\pi$, if $\zeta = \xi + i\eta$. (38) becomes

$$w(\pi i) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{w(\xi)\, d\xi}{e^{\frac{\xi}{2}} + e^{-\frac{\xi}{2}}}, \tag{44}$$

due regard having been taken to (43).

For subsequent purposes it will be handy to normalize $w(\zeta)$ by putting $w(\pi i) = +1$. Because of (42) and the other boundary conditions we then get

$$w(\pi i) = +1 = \frac{1}{\pi} \sum_{-\infty}^{+\infty} e^{(n+1)\beta\varrho} \int_{n\beta+\alpha}^{(n+1)\beta} \frac{\varphi(\xi)\, d\xi}{e^{\frac{\xi}{2}} + e^{-\frac{\xi}{2}}}, \tag{45}$$

where $\varphi(\xi) \equiv w(\xi)$ for $-\beta + \alpha \leq \xi < 0$, and then, periodically: $\varphi(\xi + \beta) \equiv \varphi(\xi)$.

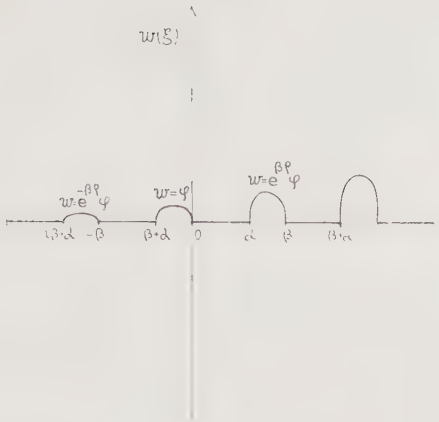

Fig. 6.

We intend to take formula (45) as a basis for the following estimations.

As regards the behaviour of $w(\zeta)$ in the neighbourhood of an S-segment, SCHWARZ' principle of symmetry shows that the normal derivative $\frac{\partial w}{\partial n}$ is positive and continuous along such a segment, except at the end-points where the gradient is infinite. In order to study the function in the vicinity of one of these end-points, a square root transformation should first be undertaken so as to bring the two sides of the segment into a straight line; in the vicinity of $\zeta = \alpha$, e. g., we find that

$$\frac{\partial w}{\partial \xi} = O\left(\frac{1}{\sqrt{\xi - \alpha}}\right).$$

It may be observed that $w(\xi)$ is concave in every interval where it is $\neq 0$, which will be obvious by application of an analogue of HADAMARD's three-circles theorem: as has been shown in section 12 $w(\xi + i\eta)$ attains its minimum, when η varies; on the ξ-axis; hence $-w(\xi) = \max_{\eta} \{-w(\xi + i\eta)\}$ is a convex function of ξ.

14. ESTIMATION OF THE ORDER FOR LARGE a, b. — In applications, it is of particular interest to know whether the

expression $\frac{\log a}{2 \log b}$ in (40) actually is the lower bound. We investigate cases with large a, b.

We keep the quotient

$$\frac{\log a}{2 \log b} = \frac{\alpha}{2\beta} = \sigma \tag{46}$$

constant and will for different σ in the interval $0 < \sigma < \frac{1}{2}$ study the order $\varrho\ (a, b)$ for increasing a and b.

We will start by studying the function $\varphi(\xi)$ in (45) somewhat closely; particularly its behaviour in the vicinity of $\xi = -\beta + \alpha$.

In order to tie in with well-known facts we prefer to revert temporarily to the z-plane, where, as we know, $\varphi(\xi) \equiv u(e^{\xi}) = u(x)$ in the interval $\frac{a}{b} \leq x \leq 1$.

Since $\frac{a}{b} = e^{\alpha - \beta} \to 0$ as a and $\beta \to \infty$ in accordance with (46), we set out by assuming that $\frac{a}{b} \leq \frac{1}{2}$.

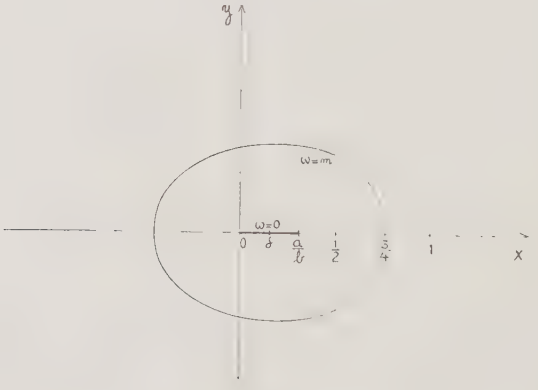

Fig. 7.

A good minorant of $u(x)$ in, for example, the interval $\left(\frac{a}{b}, \frac{3}{4}\right)$ may be obtained as follows:

We form the harmonic function $\omega(z)$ vanishing along the *whole* segment $\left(0, \frac{a}{b}\right)$ and constant $= m > 0$ on an ellipse having the

foci at $z=0$ and $z=\frac{a}{b}$ and passing through $z=\frac{3}{4}$; m being so chosen that it is less than the minimum of $u(z)$ on the ellipse. Such a choice can be made independent of a and b, provided that $\frac{a}{b}\leq\frac{1}{2}$, since the normalizing $u(-1)=+1$ and (28) give a positive lower bound for the minimum mentioned on all ellipses in question.

Hence

$$u(x)\geq\omega(x) \text{ for } \frac{a}{b}\leq x\leq\frac{3}{4}. \tag{47}$$

Now $\omega(z)$ may be identified as the logarithm of the modulus of an analytic function, well-known in the theory of conformal mapping, although the parameters are not very neat:

$$\omega(z)=K(a,b)\frac{\log\left|\frac{z-\delta}{\delta}+\sqrt{\left(\frac{z-\delta}{\delta}\right)^2-1}\right|}{\log\frac{1}{\delta}} \tag{48}$$

where we have put $\frac{a}{b}=2\delta$; the constant $K(a,b)\geq K>0$. Elementary calculation then gives

$$u(x)\geq\omega(x)\geq\frac{K}{\log\frac{1}{\delta}}\sqrt{\frac{x-2\delta}{2\delta}} \text{ for } 2\delta\leq x\leq 3\delta, \tag{49}$$

hence, since $x=e^{\xi}$, $\log\frac{1}{\delta}=\beta-\alpha+\log 2$,

$$\varphi(\xi)\geq\frac{K}{\beta-\alpha+\log 2}\sqrt{\xi-(-\beta+\alpha)} \text{ for } -\beta+\alpha\leq\xi\leq \tag{50}$$

$$\leq-\beta+\alpha+\log\frac{3}{2}.$$

The equality (45) is then transformed into an inequality by substituting for the right-hand member a quantity less than the term corresponding to $n=0$:

$$1>\frac{e^{\beta\varrho}}{2\pi}\int_{\alpha}^{\alpha+\log\frac{3}{2}}e^{-\frac{\xi}{2}}\varphi(\xi)\,d\xi. \tag{51}$$

From (50) and the periodicity of $\varphi(\xi)$ we get:

$$(52)\qquad 1 > \frac{K e^{\beta\varrho}}{2\pi(\beta-\alpha+\log 2)} \int_{\alpha}^{\alpha+\log\frac{3}{2}} e^{-\frac{\xi}{2}} \sqrt{\xi-\alpha}\, d\xi =$$

$$= \frac{K e^{\beta\varrho-\frac{\alpha}{2}}}{2\pi(\beta-\alpha+\log 2)} \int_{0}^{\log\frac{3}{2}} e^{-\frac{\tau}{2}} \sqrt{\tau}\, d\tau > \frac{e^{\beta\varrho-\frac{\alpha}{2}}}{\varkappa(\beta-\alpha)},$$

where $\varkappa$ is a positive constant independent of α and β, if, e. g., $\beta-\alpha \geq \log 2$. Consequently

$$(53)\qquad \varrho < \frac{\alpha}{2\beta} + \frac{\log \varkappa(\beta-\alpha)}{\beta}.$$

Combining this result with (40) we obtain the following theorem:

If a and b tend to infinity so that

$$\frac{\log a}{2 \log b} = \sigma$$

where σ is a number in the interval $0<\sigma<\frac{1}{2}$, then

$$(54)\qquad \lim_{a,\, b\to\infty} \varrho(a, b) = \sigma.$$

Thus 2ϱ approaches the logarithmic density of the sequence of intervals E in $(1, \infty)$.

15. THE CASE OF a AND b TENDING TO 1. — In contrariety to the investigation just performed we will now find out what happens to $\varrho(a, b)$ as a and b approach their lower bound 1, i. e. when E consists of short, closely spaced intervals.

As before we put $w(\pi i)=1$; (42) then implies that $w(\xi)<1$ for $\xi \leq 0$ and consequently $\varphi(\xi)$, being $\equiv w(\xi)$ for $-\beta+\alpha \leq \xi \leq 0$, is <1 for all ξ. In the right-hand member of (45) we then have $\frac{1}{\pi}\sum_{-\infty}^{-1} < \frac{1}{2}$, hence

$$(55)\qquad \frac{1}{2} < \frac{1}{\pi} \sum_{0}^{\infty} e^{(n+1)\beta\varrho} \int_{n\beta+\alpha}^{(n+1)\beta} \frac{\varphi(\xi)\, d\xi}{e^{\frac{\xi}{2}} + e^{-\frac{\xi}{2}}} <$$

$$< \frac{\varphi_{max}}{\pi} \sum_{0}^{\infty} e^{(n+1)\beta\varrho} \int_{n\beta+\alpha}^{(n+1)\beta} e^{-\frac{\xi}{2}}\, d\xi < \frac{\varphi_{max}}{\pi}(\beta-\alpha)\, e^{\beta\varrho-\frac{\alpha}{2}} \sum_{0}^{\infty} e^{n\beta\left(\varrho-\frac{1}{2}\right)} <$$

(since $\varrho < \frac{1}{2}$ by (39))

$$< \frac{\varphi_{max}(\beta - \alpha)\, e^{\frac{\beta-\alpha}{2}}}{\pi\left(1 - e^{\beta(\varrho-\frac{1}{2})}\right)} .$$

The inequality between the two end membra in (55) can be expressed:

$$\beta\left(\varrho - \frac{1}{2}\right) > \log\left\{1 - \frac{2}{\pi} e^{\frac{\beta-\alpha}{2}} (\beta - \alpha)\, \varphi_{max}\right\}. \tag{56}$$

Remembering that $\varphi_{max} = \max_{\xi} \varphi(\xi) < 1$, as was just pointed out, the validity of

$$\beta\left(\varrho - \frac{1}{2}\right) > -(\beta - \alpha)\, \varphi_{max} \tag{57}$$

is obvious for small values of $\beta - \alpha$; closer examination shows the truth of the statement for, e. g., $\beta - \alpha \leq \frac{1}{e}$. Thus we have

$$\varrho(a, b) > \frac{1}{2} - \left(1 - \frac{\log a}{\log b}\right) \varphi_{max} \tag{58}$$

for $\log \frac{b}{a} \leq \frac{1}{e}$.

If, b being fixed, we allow $a \to b$, i. e. to let E cover more and more of the axis, (58) yields $\varrho \to \frac{1}{2}$. This could, of course, have been surmised in advance; as $u(z)$ then converges to $u_0(z) \equiv \mathfrak{I}(\sqrt{z}) = \sqrt{r} \sin \frac{\Theta}{2}$, $z = re^{i\Theta}$, which vanishes along the whole ray $\Theta = 0$.

There is, however, another, more interesting case when $\varrho \to \frac{1}{2}$. We keep the quotient

$$\frac{\log a}{\log b} = \frac{\alpha}{\beta} = \frac{1}{\lambda} \tag{59}$$

constant, λ being a number however large, and let b and $a \to 1$. Then it seems plausible that $\varphi_{max} \to 0$; a fact which moreover is easy to prove. Since $\varphi_{max} = \max\ w(\xi)$ for $-\beta + \alpha \leq \xi \leq 0$, we look for a majorant of $w(\xi)$ in this interval. In the first place we note that, according to (41) and (28),

$$\max_{|\zeta| \leq 1} w(\zeta) < w(1 + \pi i) < M, \tag{60}$$

where the constant M is independent of α, β. We then form the harmonic function $\omega_1(\zeta)$, which is $= 0$ on the segment $(0, \alpha)$ and $= M$ on an ellipse with foci at $\zeta = 0$ and $\zeta = \alpha$, passing through $\zeta = 1$. (We reproduce part of Fig. 5; the neighbourhood of the origin for small β.)

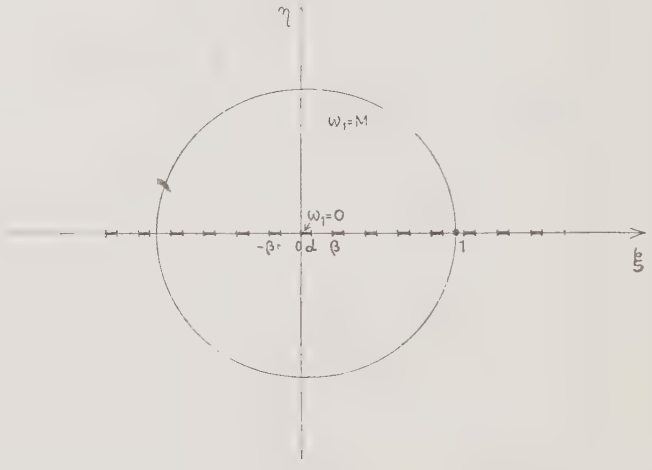

Fig. 8.

Evidently $\omega_1(\zeta)$ is a function of the same type as the one in (48), α here corresponding to 2δ. Obviously, for every fixed $\lambda > 0$, it is true that $\omega_1(\zeta) \to 0$ for $|\zeta| \leq \lambda\alpha$ as $\alpha \to 0$. Therefore

$$\varphi_{max} < \omega_1(-\beta + \alpha) < \omega_1(-\lambda\alpha) \to 0 \tag{61}$$

when $\alpha \to 0$.

The relations (58) and (61) then yield the following theorem:

If a and b decrease towards 1 so that

$$\frac{\log a}{2 \log b} = \sigma,$$

where σ is a number in the interval $0 < \sigma < \frac{1}{2}$, then

$$\lim_{a,\, b \to 1} \varrho(a, b) = \tfrac{1}{2} \tag{62}$$

is true, independent of the value of σ.

16. $u(z)$ REPRESENTED AS A LOGARITHMIC POTENTIAL. — Already at the end of section 12 we mentioned how the normal derivative $\frac{\partial u}{\partial n}$ of a positive harmonic function varies along a linear segment where u vanishes; here this question need not trouble us.

We will again use the notation in Fig. 3. We let the given expression of the radius of the circle C_m also apply to negative values of the integer m. The domain D_m — of finite connectivity — is defined as that part of D which is situated between C_m and C_{-m}. If $z_0 \subset D$, then $z_0 \subset D_m$ for $m > m_0$. We remove from D_m a circular disk $|z - z_0| \leq \varepsilon$. According to GREEN's formula for two harmonic functions $u(z)$ and $v(z)$

$$\int_{E_m + C_m + C_{-m} + C_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds = 0. \tag{63}$$

where E_m is the part of E situated in D_m and C_ε the circle around z_0. For u and v in (63) we choose our $u(z)$ and $\log \left| 1 - \frac{z_0}{z} \right|$, respectively. On C_1, for instance, $u(z)$ and its derivatives are continuous and consequently bounded; their values on C_1 determine the values on the other circles in a simple manner by means of the functional equation (36) with $\varrho < \frac{1}{2}$. It then follows that the contributions from C_m and $C_{-m} \to 0$ as $m \to \infty$, so that in the limit

$$u(z_0) = \frac{1}{\pi} \int_E \log \left| 1 - \frac{z_0}{z} \right| \frac{\partial u}{\partial n} \, dz, \tag{64}$$

or, using another notation,

$$u(z) = \int_0^\infty \log \left| 1 - \frac{z}{t} \right| d\mu(t) \tag{65}$$

where $\mu(t) = \frac{1}{\pi} \int_0^t \left(\frac{\partial u}{\partial n} \right)_{z=x} dx$ is continuous, monotonic increasing, constant in the intervals complementary to E, further, of order ϱ (a, b) and of very regular growth.

Perhaps this should be explained.

We have defined $\mu'(t)$ as the limit of $\frac{1}{\pi}\frac{\partial u}{\partial y}$ when $y \to +0$ at the point $x = t$, $y = 0$. This derivative vanishes on the intervals complementary to E, since $u(z)$ is harmonic on such intervals and also symmetric in y. Along an E-interval it is, as has been mentioned previously, positive and continuous except at the end-points; close to an end-point $t = x_0$ there is $\mu'(t) = O\left(\frac{1}{\sqrt{|t - x_0|}}\right)$. The functional equation (36) corresponds to

$$\mu'(b\,t) \equiv b^{\varrho-1}\,\mu'(t) \tag{66}$$

and to

$$\mu(b\,t) \equiv b^{\varrho}\,\mu(t) \tag{67}$$

when $\mu(0) = 0$. Similarly with (37) we also obtain

$$\mu(t) \equiv t^{\varrho}\,\chi(t) \tag{68}$$

where $\chi(t)$ is continuous and $\chi(t) \equiv \chi(bt)$.

CHAPTER IV

A REMARKABLE CLASS OF INTEGRAL FUNCTIONS

17. FROM POTENTIALS TO INTEGRAL FUNCTIONS. — Keeping the notation used in the preceding chapter and considering a positive harmonic function $u(z)$, which vanishes on the slits $E(a, b)$, we from the potential representation (65) form a new function, harmonic in D:

$$u_1(z) = \int_0^\infty \log \left| 1 - \frac{z}{t} \right| d[\mu(t)], \tag{69}$$

where $[\mu(t)]$ denotes the integer part of $\mu(t)$. Denoting the discontinuity points of $[\mu(t)]$ with $x_1, x_2, x_3, \ldots, x_n, \ldots$, we might as well write

$$u_1(z) = \sum_1^\infty \log \left| 1 - \frac{z}{x_n} \right|. \tag{70}$$

Thus $u_1(z)$ is the logarithm of the modulus of an integral function with positive zeros:

$$g(z) = \prod_1^\infty \left(1 - \frac{z}{x_n} \right). \tag{71}$$

From the discussion at the end of section 16 about the growth of $\mu(t)$ it appears that the exponent of convergence for the sequence $\{x_n\}$ and thus the order of $g(z)$ is $\varrho(a, b)$. However, we wish to learn more than this about the relation between $u(z)$ and $u_1(z) = \log |g(z)|$. To avoid meaningless complications we from D remove the points whose distance from E is ≤ 1; the remaining domain may be called D_1. Now, suppose that $z \subset D_1$. Then

$$u(z) - u_1(z) = \int_0^\infty \log \left| 1 - \frac{z}{t} \right| d\{\mu(t) - [\mu(t)]\} = \int_0^{x_1} + \int_{x_1}^\infty, \tag{72}$$

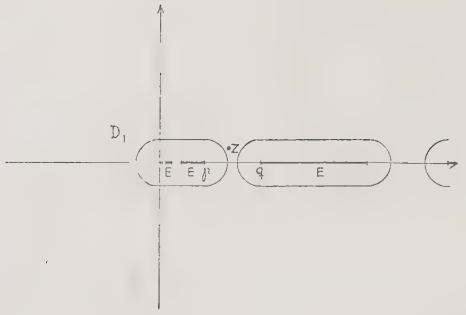

Fig. 9.

where $\mu(x_1) = 1$. We shall estimate the magnitude of these latter integrals I_1 and I_2. We have

$$I_1 = \int_0^{x_1} \log \left| 1 - \frac{z}{t} \right| d\mu(t) - \log \left| 1 - \frac{z}{x_1} \right| =$$

$$= \int_0^{x_1} \log |t - z| \, d\mu(t) - \int_0^{x_1} \log t \, d\mu(t) - \log \left| 1 - \frac{z}{x_1} \right|.$$

Since $z \subset D_1$ it follows that $|t - z| > 1$ for $t \subset E$, where $d\mu(t) > 0$, i. e.

$$\left| \int_0^{x_1} \log |t - z| \, d\mu(t) \right| < \log (x_1 + |z|) \int_0^{x_1} d\mu(t) = \log (x_1 + |z|).$$

The convergence of $\int_0^{x_1} \log t \, d\mu(t)$ is a consequence of (68).

Hence, for every $A_1 > 2$, it holds true that

$$|I_1| < A_1 \log |z| \tag{73}$$

when $z \to \infty$.

In the estimation of I_2 we first assume that the distance from z to the positive real axis is ≥ 1.

Integration by parts gives

$$(74)\qquad I_2 = \int_{x_1}^{\infty} \log\left|1-\frac{z}{t}\right| d\{\mu(t)-[\mu(t)]\} =$$

$$= -\int_{x_1}^{\infty} \{\mu(t)-[\mu(t)]\}\, d\log\left|1-\frac{z}{t}\right|,$$

since $\mu(x_1)-[\mu(x_1)]=0$. Then we have

$$(75)\qquad |I_2| < \int_{x_1}^{\infty} d\log\left|1-\frac{z}{t}\right| < \left|\log\left|1-\frac{z}{x_1}\right|\right| + 2\log|z|.$$

If the distance from z to (x_1, ∞) is <1, then z must be situated close to a complementary interval (p, q) of E. Such intervals make no contribution to the left integral in (74), therefore we might as well write $I_2 = \int_{x_1}^{p} + \int_{q}^{\infty}$, integrate by parts and obtain a result similar to (75). Since $|u(z)-u_1(z)| \le |I_1| + |I_2|$, it follows that

$$(76)\qquad |u(z)-u_1(z)| < A\log|z|$$

for every $A>5$ as $z\to\infty$ in D_1. This last restriction in the z-values is, of course, necessary; when z approaches any of the points x_n on the E-intervals we know that $u_1(z)\to-\infty$. However, the principle of harmonic majorants gives (as, by (36), $u(z)\to 0$ on the boundary of D_1 when $z\to\infty$)

$$(77)\qquad u_1(z) < u(z) + A\log|z|$$

for $A>5$ as $z\to\infty$ in any manner. This implies that

$$(78)\qquad u_1(z) - \sum_{1}^{6} \log\left|1-\frac{z}{x_n}\right| < u(z)$$

when $|z|$ is large enough. The integral function corresponding to the left-hand member of (78) we will now denote

$$(79)\qquad f(z) = \prod_{1}^{\infty}\left(1-\frac{z}{x_n}\right).$$

To the harmonic function $u(z) \subset U$ we have thus assigned an integral function $f(z)$, the class of which we will denote by Φ.

From (76) it is evident that $f(z)$, like $u(z)$, is of mean type of order $\varrho(a, b)$ and, in addition, of very regular growth. According to (78) we have that $u(z) = 0$ on the E-intervals corresponds to $|f(z)| < 1$. In the complement of E we know from (76) that $\log m(r)$ like $\min_{|z|=r} u(z)$ tends towards infinity as r^{ϱ} in the subintervals $((a+\varepsilon)b^n, (b-\varepsilon)b^n)$.

18. GENERAL CONCLUSIONS. — After these preliminaries we now have a means to estimate the precision of the theorems on integral functions in Chap. II, e. g.

$$\text{(26)} \qquad \underline{\text{log. dens.}}\ E\{m(r) > C\} \geq 1 - 2\varrho.$$

The result of the investigation of the order in section 14 implies that there exist functions $u(z) \subset U$ such that

$$\text{log. dens. } E\{\min_{|z|=r} u(z) > C\}$$

is however close to $1 - 2\varrho$. According to the preceding section, the connection between $u(z)$ and the corresponding $f(z) \subset \Phi$ is so close that the same applies to $f(z)$. Hence

the number $1 - 2\varrho$ in (26) is actually the lower bound of the density in question.

Since for $f(z)$ order and lower order are equal, we also obtain a confirmation of the exactness of (27).

Here we have not given any examples where it is known that the sign of equality holds in (26) and (27); however, it should not be difficult to perceive how such functions are constructed by combination of appropriate sequences of E-intervals.

We reproduce a sketch (very schematic!) showing how a function $f(z) \subset \Phi$ may appear for real values of $z = x + iy$:

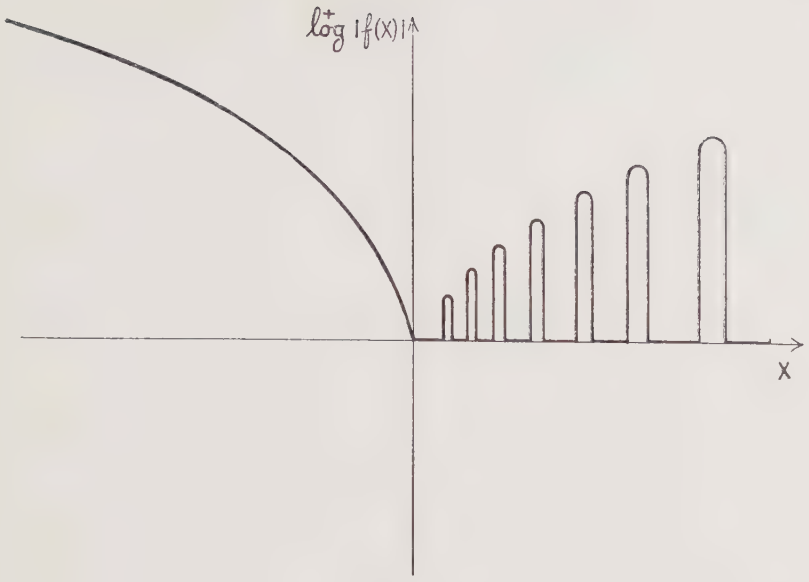

Fig. 10.

At the left we have $\log M(r)$, and at the right $\overset{+}{\log} m(r)$.

The examples where ϱ approaches $\frac{1}{2}$, i. e. the "uprights" in Fig. 10 are very narrow, seem especially interesting. As a contrast, the reader should keep in mind WIMAN's example (5) where, when $\varrho < \frac{1}{2}$, $\log m(r)$ is of the order of magnitude r^{ϱ} all along the axis except in the immediate vicinity of the zeros, but where the picture changes completely when ϱ passes $\frac{1}{2}$. Functions of class Φ prove that no such discontinuous change needs take place.

Compared to the functions (5) of perfectly regular growth, it is remarkable that the displacement of the zeros into the $E(a, b)$-intervals — a movement so slight that the functions Φ still are of very regular growth — can cause so great changes in $m(r)$. As to the distribution within the E-intervals of the zeros obtained through the jumps of $[\mu(t)]$, we content ourselves with calling attention to the fact that the earlier investigation into the properties of $\mu(t)$ tends to show that the zeros are more closely spaced at the ends of the E-intervals than elsewhere.

19. INTEGRAL FUNCTIONS BOUNDED ON A SET OF POINTS. — Functions of the class Φ can also to some extent cast light upon the problem mentioned at the end of Chap. I.

In 1931 PÓLYA posed a problem which may be formulated as follows:

Suppose that an integral function of at most order $\frac{1}{2}$, *minimal type, is bounded on the sequence* $\{n^2\}$, $n = 1, 2, 3, \ldots$. *Prove that* $f(z) \equiv$ *constant!*

Problems of this kind — evidently of the same nature as PHRAGMÉN-LINDELÖF'S theorem — have later been treated by several authors; see PFLUGER (1941) for references.

It would seem practical to treat such questions as interpolation problems; considerable difficulties, however, will be encountered unless one restricts oneself to sequences of points with some definite properties of regularity. Of course, the fact that a function is bounded on a set of points is not comparable to its having zeros at these very points.

Here we will present a few theorems without restrictive regularity conditions for the set of points whereon the function is supposed to be bounded.

We begin with the following theorem:

Suppose that $f(z)$ *is an integral function with positive zeros — if there are any — and that*

1° $|f(z)| \leq 1$ *when* z *belongs to a closed set* B *on* $z > 1$,

2° $\log M(r) \leq A(r)$, $A(r)$ *steadily increasing,*

3° $\int_1^\infty \frac{d A(r)}{\varphi(r)} < \infty$, *where* $\varphi(r)$ *is a positive, continuous, steadily increasing function such that* $\varphi(r) = o(r)$. *Let* B_φ *be the set of numbers* $x \geq 1$ *such that* $|x - r| \leq \frac{r}{\varphi(r)}$ *for at least one* $r \subset B$.

Then, if

$$\overline{\lim_{r \to \infty}} \left\{ \frac{1}{2}\, m_l B_\varphi(1, r) - \log A(r) \right\} = \infty, \tag{80}$$

it must be true that

$$f(z) \equiv \text{constant}.$$

Concerning the growth of $A(r)$ we would first point out that since

$$\int_1^\infty \frac{d\,A(r)}{\varphi(r)} > \int_1^R \frac{d\,A(r)}{\varphi(r)} > \frac{A(R) - A(1)}{\varphi(R)},$$

the condition 3° implies that $A(r) = o(r)$ as is $\varphi(r)$. Integration by parts gives then

$$\int_1^\infty \frac{\log M(r)}{r^2}\,dr < \int_1^\infty \frac{A(r)}{r^2}\,dr = A(1) + \int_1^\infty \frac{d\,A(r)}{r} < \text{const.} + \int_1^\infty \frac{d\,A(r)}{\varphi(r)},$$

i. e. $f(z)$ is of at most order 1, convergence type. Since $m_l B_\varphi(1, r) \leq \log r$, it follows from (80) that $\lim_{r\to\infty} \frac{A(r)}{r^{\frac{1}{2}}} = 0$. Thus, for a non-constant $f(z)$ it must be true that $\overline{\lim_{r\to\infty}}\, m(r) = \infty$. We will examine this case.

Those intervals, complementary to the closed set B, where $\max|f(r)| > 1$, are denoted (p_1, q_1), (p_2, q_2), ..., (p_n, q_n), ... The subset of B_φ, situated in the intervals mentioned, we call $\bar{B}_\varphi$.

We first note that between two "humps" of $m(r)$, i. e. in every interval (p_n, q_{n+1}), there must be at least one zero; viz. according to HADAMARD's three-circles theorem $\log m(r)$ is concave between two successive zeros and has consequently *one* maximum there. Denoting the number of zeros of $f(z)$ and points p_n, q_n in the interval $x \leq r$ by $n(r)$ and $N(r)$, respectively, we have, therefore

$$N(r) \leq 2\,n(r) + 1. \quad . \tag{81}$$

In well-known manner we also find a connection between $n(r)$ and $\log M(r)$; since in this case $f(z)$ is of at most order 1, convergence type, we can write $f(z) \equiv c \prod_1^\infty \left(1 - \frac{z}{x_n}\right)$, $x_n > 0$, i. e. for $r > r_0$:

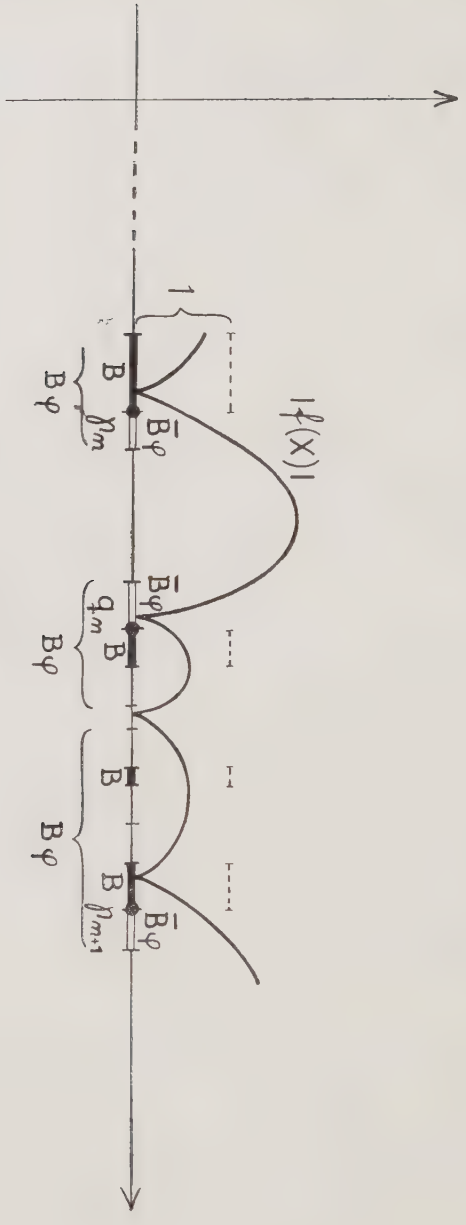
1
B
Bφ
β_m
$\bar{B}\varphi$
$|f(x)|$
$\bar{B}\varphi$
q_m
B
Bφ
B
Bφ
B
β_{m+1}
$\bar{B}\varphi$

Fig. 11.

$$\log M(r) = \log |c| + \int_0^{\infty} \log\left(1+\frac{r}{t}\right) d\,n(t) = \log |c| + \int_0^{\infty} \frac{n(t)\, r\, d\, t}{t(t+r)} >$$

$$> n(r) \int_r^{\infty} \frac{r\, d\, t}{t(t+r)} = n(r) \log 2$$

or

$$n(r) < \frac{\log M(r)}{\log 2}. \tag{82}$$

We will now estimate the logarithmic length of $\bar{B}_{\varphi}$. In any interval (p_n, q_n) we know that

$$m_l \bar{B}_{\varphi}(p_n, q_n) \leq \log\left(1+\frac{1}{\varphi(p_n)}\right) - \log\left(1-\frac{1}{\varphi(q_n)}\right), \tag{83}$$

hence we can write

$$m_l \bar{B}_{\varphi}(1, r) < \lambda_1 + \lambda_2 \int_1^r \frac{d\, N(r)}{\varphi(r)}, \tag{84}$$

where λ_1 and λ_2 are positive constants.

We study the integral in (84). Integration by parts gives

$$\int_{r_0}^{R} \frac{d\, N(r)}{\varphi(r)} = \frac{N(R)}{\varphi(R)} - \frac{N(r_0)}{\varphi(r_0)} + \int_{r_0}^{R} N(r)\, d\left(-\frac{1}{\varphi(r)}\right). \tag{85}$$

From (81), (82) and 1^0 we know that $N(r) < \frac{2}{\log 2} A(r) + 1$. Since $\varphi(r)$ is of monotonic growth the last integral is also positive which in turn implies the right to majorize (85) by substituting for $N(r)$ the majorant just obtained:

$$\int_{r_0}^{R} \frac{d\, N(r)}{\varphi(r)} < \text{const.} + \frac{2}{\log 2} \int_{r_0}^{R} \frac{d\, A(r)}{\varphi(r)}. \tag{86}$$

With regard to (84) condition 2^0 therefore leads to

$$m_l \bar{B}_{\varphi}(1, \infty) < \infty. \tag{87}$$

According to (80) it is then still true that

$$\overline{\lim_{r\to\infty}} \left\{\frac{1}{2}\, m_l\,[B_\varphi(1,r) - \bar{B}_\varphi(1,r)] - \log A(r)\right\} = \infty, \tag{88}$$

which, however, since $|f(z)| \leq 1$ in $B_\varphi - \bar{B}_\varphi$, is contrary to the BEURLING theorem (25). Thus $f(z)$ must be a constant.

As a corollary we obtain for instance the following theorem, putting $\varphi(r) = c\, r^\varrho$.

Suppose that $f(z)$ is an integral function of at most order ϱ, convergence type, which is bounded on B. Let

$$\overline{\lim_{r\to\infty}} \{m_l\, B_\varphi(1,r) - 2\,\varrho\, \log r\} > -\infty. \tag{89}$$

Then $f(z) \equiv$ constant.

From this we get, for instance, PÓLYA's theorem with "convergence type" instead of "minimal type". (Putting $\varphi(r) = \sqrt{r}$ the whole positive real axis will be covered by B_φ because $(n+1)^2 - n^2 = \sqrt{(n+1)^2} + \sqrt{n^2}$).

Still keeping $\varphi(r) = c\, r^\varrho$ we similarly obtain:

If $f(z)$ is an integral function of at most order ϱ, minimal type, which is bounded on B, then

$$\overline{\text{log. dens.}}\ B_\varphi > 2\,\varrho \tag{90}$$

implies $f(z) \equiv$ constant.

In the proof we this time get $m_l\, \bar{B}_\varphi(1,r) = o(\log r)$; i. e. we have also $\overline{\text{log. dens.}}\ \{B_\varphi - \bar{B}_\varphi\} > 2\,\varrho$ which gives the theorem.

As an application we will examine the following well-known analogy of PÓLYA's theorem:

Let $f(z)$ be an integral function of at most order ϱ, minimal type: $0 < \varrho < \frac{1}{2}$. If $f(z)$ is bounded on the sequence $x_n = n^{\frac{1}{\varrho}}$, $n = 1, 2, 3, \ldots$, then necessarily $f(z) \equiv$ constant.

Putting $\varphi(r) = \varrho\, r^\varrho$, the whole axis from a certain r-value is covered by B_φ; thus (90) is satisfied and the theorem proved.

We realize that boundedness on $\{n^{\frac{1}{\varrho}}\}$ is far more than needs to be known to draw the conclusion $f(z) \equiv$ constant. We may, for instance, formulate the following theorem:

Suppose that $f(z)$ is an integral function of at most order ϱ, minimal type: $0 < \varrho < \frac{1}{2}$. Let $f(z)$ be bounded on a sequence $a_1^{\frac{1}{\varrho}} < a_2^{\frac{1}{\varrho}} < \ldots < a_n^{\frac{1}{\varrho}} < \ldots \to \infty$, where the sequence $\{a_n\}$ is relatively dense, i. e. sup. $(a_{n+1} - a_n) < \infty$. *Then it must be true that $f(z) \equiv$ constant.*

For, the whole axis will again be covered by B_φ if the number c in $\varphi(r) = c\, r^\varrho$ is appropriately chosen.

Since we in (90) have *upper* log. dens. B_φ, it is evident that we may exclude long suites of the sequence $\{a_n^{\frac{1}{\varrho}}\}$ and still draw the same conclusion $f(z) \equiv$ constant. We here content ourselves with this observandum.

As a check on the precision of these theorems we would point out that integral functions of class Φ furnish examples of the fact that the number 2ϱ in (90) is the correct one; there exist non-constant integral functions of order less than ϱ, bounded on a set B, for which log. dens. B_φ is arbitrarily close to 2ϱ. We refer to section 18 where the E-intervals play the rôle of B. It may be of interest to note that the structure of this set is such that log. dens. B_φ remains unchanged $(= \frac{\log a}{\log b}$ for all c in $\varphi(r) = c\, r^\varrho)$.

In this connection we may observe that the metrical properties of the expanded set B_φ are decisive rather than those of the original set B. This explains the result in section 15, proving that putting $u(z) = 0$ on closely spaced intervals, whose linear and logarithmic density on $(1, \infty)$ is however small, has a similar effect on the order of $u(z)$ as putting $u(z) = 0$ all along the whole positive real axis.

CHAPTER V

A CLASSIFICATION OF BOUNDARY POINTS

20. DEFINITIONS. — In Chap. III we, for certain domains D, had reason to study the class U of positive harmonic functions $u(z)$ vanishing in the vicinity of every boundary point except one. It appeared that the boundary point in question ($z = \infty$) is such that the class U is a one-parameter family of functions; if $u_1(z) \subset U$ then every $u(z) \subset U$ may be expressed

$$u(z) \equiv c\, u_1(z)$$

where c is a positive constant.

We formulate this property of the boundary point by saying that it is *harmonically simple.*

This definition will be completed in order to cover other cases also. In so doing we restrict ourselves to boundary points which are accessible in one manner only. The existence of the class U is here taken for granted; the other boundary points of the domain in question being regular for DIRICHLET's problem, a procedure similar to that adopted in Chap. III may be used to show that this class is not void.

Suppose now that for a certain boundary point every $u(z) \subset U$ *may be expressed thus*

$$u(z) \equiv c_1 u_1(z) + c_2 u_2(z) + \ldots + c_n u_n(z), \tag{91}$$

where $u_1(z)$, $u_2(z)$, $\ldots$, $u_n(z)$ *are linearly independent functions belonging to* U, *i. e. that* U *is an* n*-dimensional family of functions.*

The boundary point is then said to be harmonically multiple and of the order n.

We will study the properties of boundary points in some cases and start with a lemma.

21. HALL'S LEMMA. — It will be convenient to avail ourselves of a theorem on harmonic measures, given by HALL

(1937), which for a half-plane corresponds to BEURLING's theorem (24) for a circle.

Let H be the upper half of the z-plane, $z = x + iy$. Let E be a set in H consisting of a finite number of JORDAN curves which together with the real axis bound a single domain D_E. Denote with E^* the circular projection of E on the positive real axis. Let $\omega(z, E, D_E)$ be the harmonic measure of E with respect to D_E, i.e. the bounded function, harmonic in D_E, which = 1 on E and = 0 for $y = 0$. Analogously we define $\omega(z, E^*, H)$ — as we know this is the visual angle, subtended by E^* at z, divided by π.

Then, it is true that

(92) $$\omega(x + iy, E, D_E) \geq h\,\omega(-|x| + iy, E^*, H),$$

where h is an absolute constant, $\frac{2}{3} < h \leq 1$.

22. SOME CASES WHEN THE BOUNDARY POINTS ARE HARMONICALLY SIMPLE. — The fact that all boundary points of a simply connected domain are harmonically simple follows from results concerning POISSON-STIELTJES integral representation of positive harmonic functions in circles and half-planes. (HERGLOTZ 1911).

However, to illustrate a method of proof we will begin by treating this simple case.

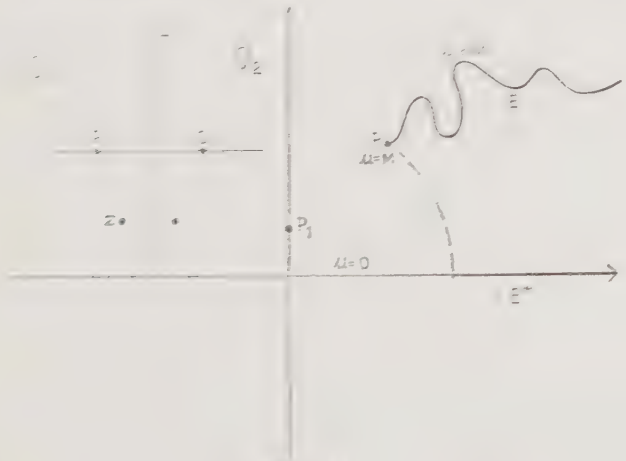

Fig. 12.

Of course, it is sufficient to consider a half-plane (we may take H in the preceding section) and let the boundary point to be examined be $z = \infty$.

To estimate the growth (the result is well-known!) of a $u(z) \subset U$ we first take an $r < 1$. Let $M(r) = \max u(r e^{i\Theta})$ when $0 \leq \Theta \leq \pi$, be attained at a point P. According to the properties of harmonic functions it is possible to choose a JORDAN curve E which originating from P extends towards infinity and along which $u(z) \geq$ the just defined number $M(r)$. Let P_1 be a fixed point in H, e. g. $z = i$.

Then, HALL's lemma gives:

$$u(P_1) > M(r) \omega(P_1, E, D_E) \geq M(r)\, h\, \omega(P_1, E^*, H).$$

However,

$$\omega(P_1, E^*, H) = \frac{1}{\pi} \operatorname{arctg} \frac{1}{r},$$

hence

$$M(r) < c_1 r \quad \text{for} \quad r > 1. \tag{93}$$

Now let R and S be two points in H with the same ordinate, else arbitrary. The mid-perpendicular L of RS divides H into the quadrants Q_1 and Q_2. For z in Q_1 we define z' in Q_2 as the symmetric point with respect to L. The harmonic function $u_1(z) = u(z) - u(z')$ vanishes on the boundary of Q_1 and is of order < 2; consequently, according to PHRAGMÉN-LINDELÖF's theorem, $u_1(z) \equiv 0$ and $u(R) = u(S)$. Hence, $u(z)$ is constant on parallels to the x-axis.

Supposing that $u(z)$ and $v(z)$ both $\subset U$, then, on a parallel $y = y_1 > 0$ it is true that $u - k_1 v = 0$, just as on the boundary line $y = 0$. Then, according to the maximum principle, $u - k_1 v \equiv 0$, Q. E. D. Thus every $u(z)$ is a positive multiple of y.

We proceed to a somewhat more general case: finitely connected domains.

Let us examine an arbitrary boundary point z_0. Again by RIEMANN's conformal mapping theorem we transform the continuum, to which z_0 belongs, into the real axis and z_0 into ∞. Our domain D may be located in the upper half-plane.

Choose an R_0 such that all other boundary contours lie within the circle $|z| = R_0$. H_0 may signify that part of D where $|z| > R_0$.

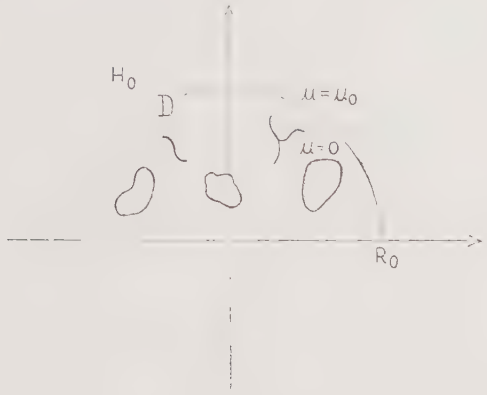

Fig. 13.

Let $u(z)$ and $v(z) \subset U$. Let $u_0(z)$ be the bounded harmonic function in H_0 which at every finite boundary point of H_0 assumes the same value as $u(z)$; similarly we define $v_0(z)$.

In the simply connected domain H_0 the function $u - u_0 \subset U$ and so does $v - v_0$, i. e. we have according to the foregoing

$$u - u_0 \equiv k(v - v_0)$$

or

$$u - k v \equiv u_0 - k v_0.$$

Since $u_0(z)$ and $v_0(z) \to 0$ as $z \to \infty$, then $u(z) - kv(z)$ is a function, harmonic in D, which tends to zero in the vicinity of *every* boundary point, consequently

$$u(z) - k v(z) \equiv 0.$$

Now it is proved that

for a domain bounded by a finite number of continua all boundary points are harmonically simple.

Furthermore, we can at once indicate cases when boundary points of infinitely connected domains must be harmonically simple:

If a boundary point z_0 is situated on a continuum and a circle $|z - z_0| = \varepsilon$ cuts off a simply connected subdomain D_ε with z_0 as boundary point, the same procedure as above may be used with the same result.

23. SOME NON-TRIVIAL CASES FOR DOMAINS OF INFINITE CONNECTIVITY. — The essential thing in the uniqueness proof in Chap. III is, that about the critical boundary point there is a sequence of closed paths C_n — uniformly approaching this point — on which it for positive harmonic functions is true that

$$\frac{u_{max}}{u_{min}} < k, \tag{94}$$

independently of n.

If the geometric conditions are favourable — if the paths C_n can be so chosen that they are at a relatively large distance from the boundary of the domain — one may from (28) obtain the inequality mentioned.

Taking a more general case than in Chap. III, let us for instance form a domain by making a finite number of radial slits — each one of them shorter than $b - 1$ — in an annula $1 \leq |z| \leq b$ and then iterate these slits by a group of transformations $T(z) = b\,z$. Exactly as when we have all slits on a radius, the boundary point $z = \infty$ is shown to be *harmonically simple.*

Now we turn our attention to a case where, in general, it is impossible to find a sequence of paths C_n such that (94) holds true.

The domain D is defined as the part of the plane outside a set of slits S on the positive real axis; we do not, as in Chap. III, restrict ourselves to the study of a particular set of intervals.

The boundary point $z = \infty$ will be the object of our examination, i. e. we intend to determine the dimension of the corresponding function family U. We begin by studying the growth of a $u(z) \subset U$.

Take an $r > 1$; let $M(r) = \max u(z)$ for $|z| = r$ be attained at a point P. Let Γ be a JORDAN curve, originating from P and extending towards infinity, along which $u(z) > M(r)$. We can

assume that Γ never crosses the positive real axis; if $u(z)$ were unsymmetric with respect to this axis we would instead estimate the magnitude of $u(z)+u(\bar{z}) \subset U$.

A minorant of $u(z)$ is obtained by cutting the plane along the positive real axis and forming the harmonic function which vanishes along this slit and is $= M(r)$ on Γ. By a transformation $\zeta = \sqrt{z}$ we map the cut plane on a half-plane and obtain a figure analogous to Fig. 12; Γ has become a curve E which originates from a point at the distance $\sqrt{r}$ from the origin. According to HALL's lemma we then get in the same manner as before

$$M(r) = \max_{|z|=r} u(z) < c\sqrt{r} \quad \text{for } r > 1. \tag{95}$$

Just as in section 12 we are then in a position to draw a number of detailed conclusions as to the growth of $u(z)$ by applying PHRAGMÉN-LINDELÖF's theorem to suitably chosen half-planes; viz. that $u(z) \equiv u(\bar{z})$, that $M(r) = u(-r)$ and — somewhat better than (95) — that

$$M(r) \leq u(-1)\sqrt{r} \text{ for } r > 1, \tag{96}$$

where the equality holds only if $u(z) \equiv k\,\mathcal{J}(\sqrt{z})$.

Now, suppose that two functions $u(z)$ and $v(z)$ both $\subset U$! We will show that their quotient must be constant.

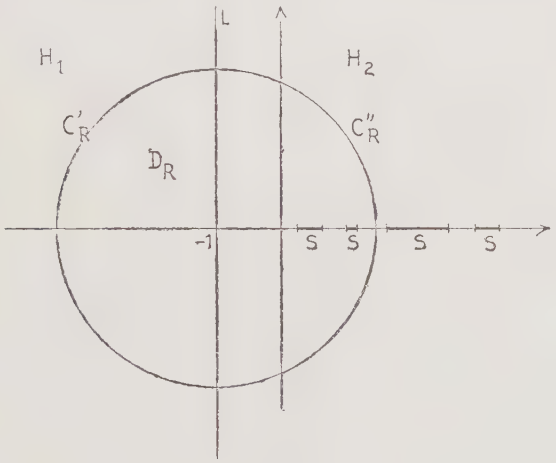

Fig. 14.

Consider first a circle $C_R : |z+1| = R$. The line $\mathscr{R}(z) = -1$ we call L; it divides the plane into H_1 and H_2, let $S \subset H_2$. C'_R and C''_R may be the semicircles in H_1 and H_2 respectively. Let D_R be the circular disk $|z+1| < R$ slit along S. We intend to use the following plausible relation:

$$\omega(-1,\ C'_R,\ D_R) \geq \omega(-1,\ C''_R,\ D_R). \tag{97}$$

This will be verified by forming $\Omega(z) \equiv \omega(z,\ C'_R,\ D_R) - \omega(z,\ C''_R,\ D_R)$ and $\Omega^*(z)$, being $+1$ on C'_R and -1 on C''_R, harmonic within C_R. Stating that $\Omega(z) - \Omega^*(z) \geq 0$ on the boundary of and therefore within C_R, we then get $\Omega(-1) \geq \geq \Omega^*(-1) = 0$, *Q. E. D.*

If $z \subset H_2$ and $z' \subset H_1$ is the symmetric point with respect to L, then it is true that $u(z) < u(z')$ if $u(z) \subset U$. (PH.-L.'s theorem).

If u_{max} and v_{min} represent extrema on C'_R we accordingly obtain

$$u(-1) < 2\, u_{max}\, \omega(-1,\ C'_R,\ D_R)$$

and

$$v(-1) > v_{min}\, \omega(-1,\ C'_R,\ D_R).$$

As we know from (28) it is possible to find a constant $k > 0$, such that on every C'_R the inequality $u_{max} < k\, u_{min}$ holds, and the same for v. We thus get

$$\frac{u_{min}}{v_{max}} > \frac{u(-1)}{2\, k^2\, v(-1)}$$

on C'_R independently of R, i. e.

$$\inf_{H_1} \frac{u}{v} = \varkappa > 0. \tag{98}$$

In H_1 therefore $u - \varkappa v \geq 0$; as this is true along L and also (with $=$) on S, the PHRAGMÉN-LINDELÖF theorem implies that it holds true in H_2 also. Hence, throughout D,

$$\frac{u}{v} \geq \varkappa > 0. \tag{99}$$

Now, supposing that $u - \varkappa v$ were not $\equiv 0$, then would $u - \varkappa v \subset U$, we again would get a relation corresponding to (99): $u - \varkappa v \geq \varkappa' v$, $\varkappa' > 0$. This is contrary to the definition of $\varkappa$ in (98), according to which $u - \varkappa v = o(v)$ as z tends to ∞ on some sequence of points. Hence we must have $u - \varkappa v \equiv 0$.

Therefore, in this case also, the boundary point $z = \infty$ is harmonically simple.

24. A CASE WHERE A BOUNDARY POINT IS HARMONICALLY MULTIPLE. — We cut the plane along intervals S on the real axis, taking S symmetric with respect to the origin. Choose a sequence of points $\{i y_n\}$, $1 < y_1 < y_2 < \ldots < y_n < \ldots \to \infty$.

As in section 9 we construct a function $u_1(z) \subset U$, the family U corresponding to the boundary point $z = \infty$, from the sequence

$$h_n(z) = \frac{G(z, i y_n)}{G(i, i y_n)},$$

where $G(z, i y_n)$ is the GREEN's function for the domain, singular at $i y_n$.

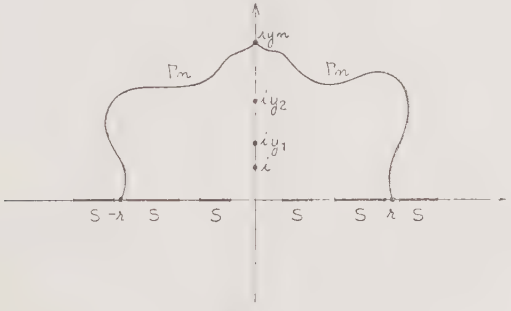

Fig. 15.

From HALL's lemma, for instance, we know that the limit function $u_1(z) = O(r)$ as $z \to \infty$. We require the same majorant of $\{h_n(z)\}$ on the real axis, independently of n; which perhaps might need some explanation. Let $M_n(r)$ be the value of $h_n(z)$ for $z = \pm r$. According to the maximum principle, in the upper half-plane curves Γ_n exist from $\pm r$ to the singular point $i y_n$, along which $h_n(z) \geq M_n(r)$. Using a well-known principle by

CARLEMAN we obtain a minorant of $h_n(z)$ in the region between Γ_n and $(-r, r)$ by forming that harmonic function which is $= 0$ on $(-r, r)$ and $= M_n(r)$ on $(-\infty, -r)$ and (r, ∞). Hence we get

$$h_n(i) = 1 > \frac{2 M_n(r)}{\pi} \operatorname{arctg} \frac{1}{r}$$

or

$$M_n(r) < \pi r \text{ for } r > 1 \text{ and every } n. \tag{100}$$

Since each $h_n(z) \to 0$ as $z \to \infty$ the values of $u_1(z)$ in the lower half-plane will be determined by the boundary values of $\{h_n(z)\}$ on the real axis. Having the majorant (100), it is only necessary to make the complementary intervals of S — called I — very small in order to obtain small values of the limit function $u_1(z)$ in the lower half-plane. If we, for instance, choose S such that the POISSON integral

$$\frac{1}{\pi} \int_I \frac{b\,|x|\,dx}{b^2 + x^2} \to 0 \text{ as } b \to \infty, \tag{101}$$

then will $u_1(z) \to 0$ as $z \to \infty$ along the negative imaginary axis. Thus $u_1(z) \not\equiv u_1(\bar{z})$. In this case, therefore, U is not a one-parameter family of functions; if we put $u_2(z) = u_1(\bar{z})$, every function of the form

$$u(z) \equiv c_1 u_1(z) + c_2 u_2(z), \tag{102}$$

c_1 and $c_2 \geq 0$, will belong to U.

If, on the other hand, we presuppose that $u(z) \subset U$ we ought to prove that it can be expressed in the form (102).

To accomplish this we first form the sum $u(z) + u(\bar{z})$. Using the same method as in the preceding section we can show that this *symmetric* function is uniquely determined but for a constant factor, i. e.

$$u(z) + u(\bar{z}) \doteq \lambda_1 (u_1(z) + u_2(z)). \tag{103}$$

On the other hand we have

$$u(z) - u(\bar{z}) \equiv \lambda_2 (u_1(z) - u_2(z)); \tag{104}$$

these differences are equal to zero on the real axis and our knowledge of their growth therefore yields (104), analogously to what was shown in section 22.

Addition of (103) and (104) gives $u(z)$ represented as in (102).

It is now proved that, in this case, the boundary point $z = \infty$ *is harmonically multiple of order 2.*

To obtain a boundary point of this type, it is not necessary to let the "gaps" I take up so small a part of the axis as required by (101). From the foregoing we may infer that one might just as well take short, closely spaced S-intervals; letting the I-intervals cover the major portion of the axis.

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